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STABILITY OF CONTROLLABLE ELASTIC DISTRIBUTED SYSTEMS*

E.M. POTAPENKO

Simple criteria for the observability of elastic systems are established. Theorems are proved that enable one to determine whether distributed controllable systems, whether linear or non-linear, are asymptotically stable, by examining a model not involving elasticity. The results are obtained without truncation of elastic modes.

Elasticity in the structure of controlled objects may modify the characteristics of the system to such a degree that a control system developed without allowing for elasticity, or allowing for only a few modes of elastic vibration, does not guarantee stability of the real physical system, since it may lead to instability in the omitted modes. A further complication is the approximate nature of dynamical schemes for real controlled objects; the only more or less reliable parameters are those relating to the lowest elastic modes. In addition, the output characteristics of the various sensors and actuating elements are usually non-linear and governed by differential equations. Hence the importance of developing methods for the synthesis and analysis of non-linear control systems for objects with inaccurately specified characteristics, in such a way as to guarantee asymptotic stability of the state of equilibrium of the full system without truncation of elastic vibratory modes. This problem will be solved in the present paper.

1. Equations of motion. Consider a rigid body E_0 of mass m_r, attached to which are elastic elements $E_1, E_2, \ldots, E_N$ of masses $m_1, m_2, \ldots, m_N$, respectively. The space S occupied by the entire system and the mass m of the entire system are defined by

$$S = E_0 + E_1 + \ldots + E_N,\ m = m_r + m_e = m_r + m_1 + m_2 + \ldots + m_N$$

It is assumed that the elastic elements are rigidly attached and experience small elastic deformations relative to their undeformed states at every point (other than the attachment points).

Let $Oxyz$ be an orthogonal reference frame attached to the rigid body. Let $\mathbf{v}_0$ and $\boldsymbol{\theta}$ be, respectively, the linear displacement vector of O and the small angle through which the body rotates about O relative to the inertial frame; let r be the vector of any point of the mechanical system in the nominal (undeformed) state relative to the frame $Oxyz$, and $u(r)$ the elastic deformation of the system at the point r. Then the displacement of any point of the system relative to the inertial frame is given by

$$v(r) = v_0 - r_*\theta + u(r) \tag{1.1}$$

written in matrix notation in terms of the projections of vectors on the axes of the frame $Oxyz$. Here and below a_* is a skew-symmetric matrix whose elements are the projections of the vectors a, and a_*b is the matrix representation of the vector product of vectors a and b.

The absolute velocity vector is defined by

$$v^\cdot(r) = v_0^\cdot - r_*\theta^\cdot + u^\cdot(r) \tag{1.2}$$

The momentum and angular momentum vectors are defined by

$$K = \int_m v^\cdot\, dm, \quad Q = \int_m r_* v^\cdot\, dm \tag{1.3}$$

Substitution of (1.2) into (1.3) gives

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$$K = \int_m (v_0^{\cdot} - r_* \theta^{\cdot} + u^{\cdot})\, dm, \quad Q = \int_m r_* (v_0^{\cdot} - r_* \theta^{\cdot} + u^{\cdot})\, dm \tag{1.4}$$

We have

$$\int_m r_*\, dm = r_{c*} m, \quad \int_m r_* r_*\, dm = -J_0$$

where r_c is the position of the centre of mass of the entire system in the $Oxyz$ frame, and J_0 is the matrix of moments of inertia of the system relative to the axes of the $Oxyz$ frame. These relations give

$$K = mv_0^{\cdot} - mr_{c*}\theta^{\cdot} + \int_{m_e} u^{\cdot}\, dm$$
$$Q_0 = J_0\theta^{\cdot} - mr_{c*}v_0^{\cdot} + \int_{m_e} r_* u^{\cdot}\, dm$$

The laws governing the variation of momentum and angular momentum yield the equations

$$mv_0^{\cdot\cdot} - mr_{c*}\theta^{\cdot\cdot} + \int_{m_e} u^{\cdot\cdot}\, dm = F_0 \quad J_0\theta^{\cdot\cdot} + mr_{c*}v_0^{\cdot\cdot} + \int_{m_e} r_* u^{\cdot\cdot}\, dm = G_0 \tag{1.5}$$

Here F_0 is a vector representing the external forces acting on the system at O, and G_0 is the moment of these forces about O. It is assumed that the forces and moments are applied only to the rigid body. Then the inertial force per unit volume of the elastic part of the system is balanced by elastic forces $-L(u)$ and internal damping forces, which are approximately described by the expressed $-Qu^{\cdot}$. By (1.2),

$$\rho(r)\,[v_0^{\cdot\cdot} - r_*\theta^{\cdot\cdot} + u^{\cdot\cdot}(r)] + Q(r)\,u^{\cdot} + L(u) = 0 \tag{1.6}$$
$$Q(r) > 0$$

where $\rho(r)$ is the density of the elastic part of the system and $L(u)$ is a differential vector operator representing the elastic forces, whose form depends on that of the elastic elements. In particular, for a flexible beam experiencing plane bending

$$L(u) = \frac{\partial^2}{\partial x^2} EI(x) \frac{\partial^2 u}{\partial x^2}$$

where $EI(x)$ is the flexural rigidity and x coordinate reckoned along the beam.

Multiplying the first equation of (1.5) on the left by $v_0^{\cdot T}$, the second by $\theta^{\cdot T}$, and Eq.(1.6) by $u^{\cdot T}$ and then adding all three equations, we obtain the law governing the overall mechanical energy of the system:

$$\frac{d}{dt}\Big\{\Big[\frac{1}{2} mv_0^{\cdot T}v_0^{\cdot} + \frac{1}{2}\theta^{\cdot T}J_0\theta^{\cdot} - mv_0^{\cdot T}r_{c*}\theta^{\cdot} + v_0^{\cdot T}\int_{m_e} u^{\cdot}\, dm + \theta^{\cdot T}\int_{m_e} r_* u^{\cdot}\, dm +$$
$$\frac{1}{2}\int_{m_e} u^{\cdot T}u^{\cdot}\, dm\Big] + \frac{1}{2}\int_{S_e} u^T L(u)\, dS\Big\} = -\int_{S_e} u^{\cdot T}Qu^{\cdot}\, dS + v_0^{\cdot T}F_0 + \theta^{\cdot T}G_0 \tag{1.7}$$

The bracketed expression is the kinetic energy T. On the other hand, by (1.2),

$$T = {}^1\!/_2 \int_m (v_0^{\cdot} - r_*\theta^{\cdot} + u^{\cdot})^T (v_0^{\cdot} - r_*\theta^{\cdot} + u^{\cdot})\, dm$$

Evaluating the integral with respect to the mass of the rigid body, we can write

$$T = {}^1\!/_2 m_r v_0^{\cdot T} v_0^{\cdot} + {}^1\!/_2 \theta^{\cdot T} J_{r0}\theta^{\cdot} + m_r v_0^{\cdot T}\theta_*^{\cdot} r_{rc} +$$
$${}^1\!/_2 \int_{m_e} (v_0^{\cdot} - r_*\theta^{\cdot} + u^{\cdot})^T (v_0^{\cdot} - r_*\theta^{\cdot} + u^{\cdot})\, dm$$

where J_{r0} is the matrix of moments of inertia of the rigid body relative to the frame $Oxyz$, and r_{rc} the radius-vector of the centre of mass of the rigid body in the frame $Oxyz$.

The overall mechanical energy of the system is

$$V' = {}^1/_2 m_r v_0^{\cdot T} v_0^{\cdot} + {}^1/_2 \theta^{\cdot T} J_{r0} \theta^{\cdot} + m_r v_0^{\cdot T} \theta_*^{\cdot} r_{rc} + \\ {}^1/_2 \int\limits_{m_e} (v_0^{\cdot} - r_* \theta^{\cdot} + u^{\cdot})^T (v_0^{\cdot} - r_* \theta^{\cdot} + u^{\cdot})\, dm + {}^1/_2 \int\limits_{S_e} u^T L(u)\, dS \tag{1.8}$$

By (1.7),

$$V'^{\cdot} = v_0^{\cdot T} F_0 + \theta^{\cdot T} G_0 - \int\limits_{S_e} u^{\cdot T} Q u^{\cdot}\, dS \tag{1.9}$$

System (1.5) and (1.6) consists of ordinary differential equations and a partial differential equation (since $L(u)$ is a partial differential operator). This system will now be replaced by an equivalent infinite system of ordinary differential equations. Define a transformation

$$u_n(r, t) = \Phi_n(r)\, q_n(t), \quad n = 1, 2, \ldots, N \tag{1.10}$$

where n is the number of the elastic element, $\Phi_n(r)$ is a $(3 \times \infty)$-matrix of normalized admissible functions satisfying orthogonality conditions, and $q_n(t)$ is an infinite vector of generalized "elastic" coordinates. Substituting (1.10) into Eqs.(1.5) and (1.6) gives /1/

$$\begin{gathered} m v_0^{\cdot\cdot} - m r_{c*} \theta^{\cdot\cdot} + P^T q^{\cdot\cdot} = F_0 \\ J_0 \theta^{\cdot\cdot} + m r_{c*} v_0^{\cdot\cdot} + H^T q^{\cdot\cdot} = G_0 \\ P v_0^{\cdot\cdot} + H\theta^{\cdot\cdot} + q^{\cdot\cdot} + R q^{\cdot} + \Omega^2 q = 0 \\ P = [P_1^T, \ldots, P_N^T]^T, \quad H = [H_1^T, \ldots, H_N^T]^T, \quad \Omega^2 = \mathrm{diag}[\Omega_1^2, \ldots, \Omega_N^2], \\ R = \mathrm{diag}[R_1, \ldots, R_N] \\ P_n^T = \int\limits_{m_n} \Phi_n\, dm, \quad H_n^T = \int\limits_{m_n} r_* \Phi_n\, dm \\ \Omega_n^2 = \mathrm{diag}[\Omega_{n1}^2, \Omega_{n2}^2, \ldots], \quad R_n = \int\limits_{E_n} \Phi_n^T Q \Phi_n\, dE_n > 0 \end{gathered} \tag{1.11}$$

where P_n, H_n are the matrices of influence coefficients, Ω_n is the matrix of natural frequencies of the elastic elements (assuming the rigid body to be fixed), R_n the matrix of natural damping coefficients.

As admissible functions one can take the eigenfunctions obtained by solving the boundary-value problem with fixed rigid body /1-5/, i.e., the equation $\rho(r) u^{\cdot\cdot} + L(u) = 0$.

If the functions $\Phi_n(r)$ form a complete system, it follows from Parseval's theorem /1/ that

$$P^T P = m_e E_3, \quad H^T P = m_e r_{ce*}, \quad H^T H = J_e \tag{1.12}$$

where the index e refers to the elastic body, $E_3 = \mathrm{diag}[111]$, r_{ce} is the vector of the centre of mass of the elastic part of the system in the $Oxyz$ frame, and J_e is the matrix of moments of inertia of the elastic part of the system in the $Oxyz$ frame.

Multiplying the first equation of system (1.11) on the left by $v_0^{\cdot T}$, the second by $\theta^{\cdot T}$, and the third by $q^{\cdot T}$ and then adding the resulting equations, we obtain the law governing the overall energy:

$$V'^{\cdot} = v_0^{\cdot T} F_0 + \theta^{\cdot T} G_0 - q^{\cdot T} R q^{\cdot} \tag{1.13}$$

In view of (1.12), we can write

$$2V' = m_r v_0^{\cdot T} v_0^{\cdot} + \theta^{\cdot T} J_{r0} \theta^{\cdot} + 2 m_r v_0^{\cdot T} \theta_*^{\cdot} r_{rc} + q^T \Omega^2 q + \\ (P v_0^{\cdot} + H\theta^{\cdot} + q^{\cdot})^T (P v_0^{\cdot} + H\theta^{\cdot} + q) \tag{1.14}$$

Eqs.(1.5) and (1.11) may be considerably simplified if the $Oxyz$ frame is aligned with the principal central axes of the system in the normal state.

2. ***Observability.*** Suppose tht measuring devices (sensors) are placed on the rigid body of our mechanical system, producing a vector of output signals

$$y = C\, [v^T \theta^T]^T \tag{2.1}$$

where the inverse matrix C^{-1} exists. To analyse the observability of system (1.11), (2.1), we use the criterion according to which a system is observable if and only if the only solution compatible with output signals zero is the trivial solution. Now, if $0 \equiv C\,[v^T\theta^T]^T$, then $[v^T\theta^T]^T \equiv [v^{\cdot T}\theta^{\cdot T}]^T \equiv [v^{\cdot\cdot T}\theta^{\cdot\cdot T}]^T \equiv 0$. Using these identifies, we can write the system as

$$P^T q^{\cdot\cdot} = 0, \quad H^T q^{\cdot\cdot} = 0, \quad q^{\cdot\cdot} + Rq^{\cdot} + \Omega^2 q = 0 \tag{2.2}$$

This equation is equivalent to an infinite system of oscillators. In cases of practical importance these oscillators are weakly damped and their characteristic equations have complex roots. Let us assume that there are no multiple roots. This is the case, at least, when the matrix Ω has no multiple frequencies and either $R = 0$ or R is a diagonal matrix. Then the solution of the last equation in (2.2) is an infinite sequence of linearly independent functions, forming the vector q. The coordinates of the vector $q^{\cdot\cdot}$ are also linearly independent. The first two equations in system (2.2) are equivalent to a system of six homogeneous algebraic equations. Since the coordinates of $q^{\cdot\cdot}$ are linearly independent, it follows that if not all the elements of the corresponding rows of the matrices P^T and H^T vanish, each of the six equations is possible only if $q \equiv 0$. This implies the following observability criterion.

Criterion 1. For weakly damped elastic elements, if the matrix Ω has no multiple frequencies and $R = 0$ or $R = \text{diag}$, the only possible non-observable coordinates are those for which the corresponding rows of P and H vanish simultaneously. If at least one element of these rows does not vanish, the corresponding coordinate is observable.

A special case of this criterion was established in /2/.

Now suppose that the third equation in (2.2) has equal roots, totalling s pairs. This is possible if the matrix Ω has s equal frequencies. In that case one has the following

Criterion 2. If the third equation in (2.2) has equal roots, totalling s pairs, then the corresponding coordinates of the vector q are observable if and only if

$$\text{rank}\,[P_s H_s]_{s\times 6} = s \tag{2.3}$$

where P_s and H_s are the submatrices of P and H corresponding to the coordinates of q with multiple roots. In this case $s \leqslant 6$, because the matrix $[P_s H_s]$ in this system has six columns, so its rank cannot exceed six. If there are several sets of multiple roots, formula (2.3) will be used to check each set separately for observability.

Criterion 1 was etablished above for a system more general than those considered in /3, 4/ and of a rather different form. It can be shown that these criteria are equivalent.

Condition (2.3) is also valid in the case of multiplicity equal to unity. Let i denote the index of the coordinate in question. Then for an observable coordinate

$$\text{rank}\,[P_i H_i] = \text{rank}\,[P_i H_i]\,[P_i H_i]^T = \text{rank}\,[P_i H_i]^T\,[P_i H_i] = 1$$

Combining these equalities for $i = 1, 2, \ldots$, we can write the observability conditions

$$\begin{gathered} P_i P_i^T + H_i H_i^T \neq 0 \\ \begin{bmatrix} P_i^T P_i & P_i^T H_i \\ H_i^T P_i & H_i^T H_i \end{bmatrix} > 0, \quad i = 1, 2, \ldots \end{gathered} \tag{2.4}$$

which generalize the criteria of /3, 4/. Criterion 1 is simpler than (2.4), since it requires no mathematical operations.

It can be shown that if the system is controllable without allowing for elasticity, the conditions for controllability, allowing for elasticity, are precisely the same as the conditions for observability.

3. ***Equations of the controller.*** Let us assume that the sensors and actuating mechanisms are situated on the rigid body. The operation of the actuators is described by the equations

$$\begin{aligned} F_0^{\cdot} + \alpha F_0 &= f(g, p, d, s, w) \\ G_0^{\cdot} + \beta G_0 &= \varphi(g, p, d, s, w) \end{aligned} \tag{3.1}$$

Here α, β are positive definite matrices characterizing the time constants of the actuators, and $f(\cdot)$ and $\varphi(\cdot)$ are odd functions describing the non-linearities of the actuators and of the control arguments; it is assumed that $f(0, 0, 0, 0, 0) = \varphi(0, 0, 0, 0, 0) = 0$; g, p, d, s are vectors representing the output signals of the sensors:

$$\begin{aligned} g^{\cdot} + \gamma g = \psi(v_0),\ p^{\cdot} + \delta p = \omega(v_0^{\cdot}) \\ d^{\cdot} + \eta d = \chi(\theta),\ s^{\cdot} + \varepsilon s = \zeta(\theta^{\cdot}) \end{aligned} \tag{3.2}$$

where $\psi(v_0)$, $\omega(v_0^{\cdot})$, $\chi(\theta)$, $\zeta(\theta^{\cdot})$ are odd vector-valued functions characterizing the non-linearities of the sensors, $\psi(0) = \omega(0) = \chi(0) = \zeta(0) = 0$, $\gamma, \delta, \eta, \varepsilon$ are positive definite matrices describing the time constants of the sensors, and w is the state vector of a compensator, defined by the equation

$$w^{\cdot} = Fw + K_1 g + K_2 p + K_3 d + K_4 s \tag{3.3}$$

Here $F, K_1, \ldots, K_4$ are constant matrices.

The functions $f, \varphi, \psi, \omega, \chi, \zeta$ are assumed to be continuous and such that the set of Eqs. (1.11), (3.1)-(3.3) has a unique solution.

4. Stability of motion. If $m_r > 0, J_{r0} > 0$, the sum of the first three terms on the right of (1.14) is a positive definite function of the coordinates of the vectors $v_0^{\cdot}, \theta^{\cdot}$, as a quadratic constituent of the kinetic energy of the body. If the elastic elements are attached to the rigid body as stipulated above, $q^T\Omega^2 q$ is a positive definite function of q. We can therefore state that if $m_r > 0, J_{r0} > 0$ then V' as determined by (1.14) is a positive definite function of the coordinates of system (1.11). This is no longer true if $m_r = 0$ and/or $J_{r0} = 0$.

In order to conclude that the functional (1.8) is positive definite, it will suffice to show that its density is positive definite /6, 7/. To that end, we attach a reference frame to the n-th elastic element in its undeformed state, say $Ox_n y_n z_n$, chosen in such a way that elastic displacements along the axes are independent. By Rayleigh's relation for the potential energy of the elastic forces, we can write /6, 7/

$$\int_{S_n} u^T L(u)\, dS = \int_{S_n} u_n^T L(u_n)\, dS \geqslant \int_{S_n} \rho(r)\, u_n^T \Omega_{1n}^2 u_n\, dS,$$
$$\Omega_{1n}^2 = \operatorname{diag}[\Omega_{1x}^2, \Omega_{1y}^2, \Omega_{1z}^2] \tag{4.1}$$

where $\Omega_{1x}, \Omega_{1y}, \Omega_{1z}$ are the least eigenvalues corresponding to elastic vibrations along the axes $Ox_n, Oy_n; Oz_n$, the rigid body being fixed; the column-matrix u_n is expressed in terms of its projections on the $Ox_n y_n z_n$ axes. The relation between u and u_n is defined by the matrix of direction cosines between the frames $Oxyz$ and $Ox_n y_n z_n$. Similar relationships hold for all the elastic elements. Thus, using expression (1.8) for the density V_d' of the functional, we can write

$$2V_d' = m_e^{-1}[m_r v_0^{\cdot T} v_0^{\cdot} + \theta^{\cdot T} J_{r0}\theta^{\cdot} + 2m_r v_0^{\cdot T}\theta_*^{\cdot} r_{rc}] +$$
$$(v_0^{\cdot} - r_*\theta^{\cdot} + u^{\cdot})^T (v_0^{\cdot} - r_*\theta^{\cdot} + u^{\cdot}) + \rho^{-1}(r)\, u^T L(u) \tag{4.2}$$

Using (4.1), we can verify that the density (4.2) is positive definite provided that $m_r > 0$, $J_{r0} > 0$. Under these conditions, therefore, the functional (1.8) is also positive definite.

Together with system (1.5), (1.6) and (1.11), we shall also consider the system with the elastic terms neglected:

$$mv_0^{\cdot\cdot} - mr_{c*}\theta^{\cdot\cdot} = F_0, \quad J_0\theta^{\cdot\cdot} + mr_{c*}v^{\cdot\cdot} = G_0 \tag{4.3}$$

for which

$$V_0'^{\cdot} = v_0^{\cdot T} F_0 + \theta^{\cdot T} G_0 \tag{4.4}$$
$$2V_0' = mv_0^{\cdot T} v_0^{\cdot} + \theta^{\cdot T} J_0 \theta^{\cdot} - 2mv_0^{\cdot} r_{c*}\theta^{\cdot} \tag{4.5}$$

Theorem 1. If there exists a positive definite function V_0 for the system (4.3), (3.1)-(3.3) with the elastic terms neglected, in which the vectors $v_0^{\cdot}, \theta^{\cdot}$ occur only in the term V_0' of (4.5), and the total derivative $V_0^{\cdot}$ with respect to time along trajectories of system (4.3), (3.1)-(3.3) is negative definite, then the equilibrium state of system (1.5), (1.6), (3.1)-(3.3) or system (1.11), (3.1)-(3.3), which describe the system including the elastic terms, are asymptotically stable provided that $m_r > 0$, $J_{r0} > 0$.

Proof for system (1.5), (1.6), (3.1)-(3.3). Considering V_0, replace the term V_0' of (4.5) by V' as in (1.8); this gives a positive definite functional V for system (1.5), (1.6), (3.1)-(3.3). The functional $V^{\cdot}$ is negative definite (to see this, compare (1.9) and (4.4)) and vanishes at $v_0 = v_0^{\cdot} = \theta = \theta^{\cdot} = 0$, $u^{\cdot} = 0$. If $V^{\cdot} \equiv 0$, then $v_0^{\cdot\cdot} \equiv \theta^{\cdot\cdot} \equiv 0$, $u^{\cdot} \equiv u^{\cdot\cdot} \equiv 0$, and this, as follows from Eqs.(1.5), (1.6), corresponds to a fixed position of the rigid body, relative to which the elastic part is at rest. If the elastic elements are rigidly attached, the only point at which $V^{\cdot} \equiv 0$ is the point at which $u \equiv u^{\cdot} \equiv 0$, i.e., the equilibrium point. It then follows from a theorem of /7/ that the equilibrium state of our system is indeed asymptotically stable.

Proof for system (1.11), (3.1)-(3.3). Considering V_0, replace the term V_0' of (4.5) by V' as in (1.14); this gives a positive definite function V' for this system. The function V for this system. The function $V^{\cdot}$ of (1.13) is negative definite in this case. If the elastic elements involve natural damping, the proof proceeds as before. If the weak damping is negligible, we must set $R = 0$ in the last equation of (1.11) and in (1.13).

Under what conditions is the system asymptotically stable? Comparing $V^{\cdot\cdot}$ (1.13) and $V_0'^{\cdot}$ (4.4), with due consideration of (3.1)-(3.3), we see that $V^{\cdot} \equiv 0$ if $F_0 = G_0 = v_0 = v_0^{\cdot} = \theta = \theta^{\cdot} = 0$, so that $v^{\cdot\cdot} = \theta^{\cdot\cdot} = 0$. Substitution of these identities into Eq.(1.11) gives system (2.2) with $R = 0$. Hence it follows that the system will have no complete trajectories other than the trivial one with $V^{\cdot} \equiv 0$, if and only if it is observable. Hence, by the Barbashin-Krasovskii Theorem, the trivial solution of system (1.11), (3.1)-(3.3) is asymptotically stable.

Theorem 2. If there exists a positive definite function V_0 for system (4.3), (3.1)-(3.3), with the elastic terms neglected, in which the vectors $v_0^{\cdot}, \theta^{\cdot}$ occur only in the term V_0' of (4.5), and the total derivative $V_0^{\cdot}$ with respect to time along the trajectories of the system is negative definite, and moreover the set $V_0^{\cdot} = 0$ for the system including elasticity contains no complete trajectories other than the point zero, then the equilibrium state of system (1.5), (1.6), (3.1)-(3.3) or system (1.11), (3.1)-(3.3) with elasticity included is asymptotically stable provided that $m_r > 0,\ J_{r0} > 0$.

The proof is similar to that of Theorem 1.

Remark 1. The conditions of the theorems will be satisfied if the system with elastic terms ignored is asymptotically stable.

Remark 2. The statement of the theorem is unchanged if the operation of the sensors and/or actuators is governed by equations other than (3.1)-(3.3), provided that these equations satisfy the conditions of Lyapunov's second method.

Remark 3. Whether the system is stable or unstable does not depend on the position of the frame $Oxyz$. However, the relationship between the terms in V_0' of (4.5) does depend on the choice of frame. Thus our condition on the structure of the function V_0 is not too restrictive.

Remark 4. Our observability and stability results are easily extended to discrete models of elastic systems, irrespective of the specific method of discretization adopted.

Remark 5. Synthesis of control systems utilizing these theorems yields control systems that are more robust relative to the parameters of the elastic elements, and also relative to the inertia-mass characteristics of the controlled object.

In the case of linear systems, the determination of quadratic functions V_0 and $V_0^{\cdot}$ may be formalized through the use of Lyapunov's equation. To do so, one reduces system (4.3), (3.1)-(3.3) to Cauchy form $x^{\cdot} = Ax$ (x is the state vector of the system) and tries to find functions V_0 and $V_0^{\cdot}$ in the form $V_0 = x^TBx,\ V_0^{\cdot} = x^TCx$. Lyapunov's equation is then $A^TB + BA = C$. The specific structure of the function V_0 imposes restrictions on the structure of the matrix B. Another method for constructing functions V_0 and $V_0^{\cdot}$ will be described below, in the context of an example.

It is striking that application of the above theorems requires no knowledge of a model of the system including elasticity. The theorems reduce stability analysis for the solutions of systems of infinite order to stability analysis of systems of lower order. A similar route was adopted in /8/, using the root hodograph method, for a regulator of a special form; the question was raised there as to whether this simplified approach is applicable to a broader class of regulators. Our theorems show that this is indeed the case.

5. *Example. Synthesis of a dynamic regulator without allowing for elasticity, ensuring asymptotic stability of the system with elasticity.* We consider the one-dimensional motion of a rigid body, where the coordinate θ and velocity $\theta^{\cdot}$ are measurable. The governing equations of the controlled object and of the regulator are

$$\begin{gathered} J\theta^{\cdot\cdot} + \sum_{i=1}^{\infty} H_i q_i^{\cdot\cdot} = G, \quad H_i\theta^{\cdot\cdot} + q_i^{\cdot\cdot} + \Omega_i^2 q_i = 0 \\ G = -k_1\theta^{\cdot} - k_2\theta - k_3 u_0 - Bu \\ u_0^{\cdot} = -fu_0 + \theta, \quad u^{\cdot} = Fu + K\theta \end{gathered} \tag{5.1}$$

where k_1, k_2, k_3, f are constant coefficients, and B, F, K are constant matrices. For small f the variable u_0 is close to $\int \theta dt$, so that the system is practically astatic.

We change to the new variables

$$u_0 - f^{-1}\theta = z_0, \quad u + F^{-1}K\theta = z \tag{5.2}$$

(assuming that F is a non-singular matrix). In terms of these variables system (5.1), with elastic terms ignored, becomes

$$J\theta^{\cdot\cdot} = k_1\theta^{\cdot} - (k_2 + k_3 f^{-1} - BF^{-1}K)\theta - k_3 z_0 + Bz \tag{5.3}$$
$$z_0^{\cdot} = -fz_0 - f^{-1}\theta^{\cdot}, \quad z^{\cdot} = Fz + F^{-1}K\theta^{\cdot}$$

Multiply the first equation in (5.3) by $\theta^{\cdot}$, the second by $\rho_0 z_0$, and the third by $z^T H$, where $\rho_0 > 0$ is a scalar and H is a symmetric matrix; add the results together. This gives

$$2V_0 = J\theta^{\cdot 2} + (k_2 + k_3 f^{-1} - BF^{-1}K)\,\theta^2 + \rho_0 z_0^2 + z^T Hz$$
$$V_0^{\cdot} = -k_1\theta^{\cdot 2} - \rho_0 f z_0^2 + z^T HFz - (k_3 + \rho_0 f^{-1})\,\theta^{\cdot} z_0 + \theta^{\cdot} z^T\,(HF^{-1}K - B^T)$$

If

$$k_2 + k_3 f^{-1} - BF^{-1}K > 0, \ \rho_0 > 0, \ H > 0 \tag{5.4}$$

then the function V_0 is positive definite. Sufficient conditions for $V_0^{\cdot}$ to be negative definite are, for example,

$$k_1 > 0, f > 0, \quad \rho_0 > 0, \quad 2\sqrt{k_1\rho_0 f} > |\, k_3 + \rho_0 f^{-1}\,| \tag{5.5}$$
$$HF^{-1}K - B^T = 0, \ HF > 0$$

If $H_i \neq 0$ and Conditions (5.4), (5.5) are satisfied, we are in the situation of Theorem 2 and so system (5.1) is asymptotically stable.

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